retardation of this wave and a certain magnification of its amplitude. Passing over a trough ($b = -0.3$) the bow wave accelerates and its amplitude diminishes as compared with the case of a level bottom. It is interesting to note that the second hump of the inflowing wave is described sufficiently well by the solution of the problem for a level bottom even if it is directly above an obstacle but is not described by the solution of the corresponding problem for an infinite fluid. This indicates that the influence of bottom roughness on the free boundary shape appears not directly above an obstacle but is shifted in the direction of perturbation propagation.

The free surface shape for a bottom profile described by the function $h(x) = 1 - b \times [1 + \tanh \beta(x - x_0)]/2$ at the time $t = 12$ is represented in Fig. 4. Here $x_0 = 7$, $\beta = 0.33$, $b = \pm 0.3$; -1 (curve 1-3), and the front boundaries are $x_f = 11.21$, 12.75, 14.46. All the assertions referring to the localized roughness are valid even in the case of a smooth passage from one depth to another (see Fig. 4). The steepness of the wave will be smaller during emergence of the bow wave in the large depth domain, the greater the drop in depth.

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SOLUTION OF THE PROBLEM OF IDEAL FLUID FLOW IN THE NEIGHBORHOOD OF BODY AND WING APICES

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UDC 532.5

For a uniformly accurate description of ideal fluid flow around three-dimensional bodies, the nature of its asymptotic behavior must be known in the neighborhood of the singular points that are the body and wing apices, for example. It is known that in the neighborhood of sharp apices the flow potential depends as a power-law on the distance to the apex.

An algorithm is proposed in this paper to solve eigenvalue problems by using the method of "vortex frames" and a panel method that permit finding the eigenvalues of the exponent and eigenfunctions of the problem. Examples are presented of application of the proposed method for problems of the flow around delta wing apices and apices of a body in the form of a circular cone that have an exact solution (the problems reduce to solving an ordinary differential equation). A comparison is given between the results of computations and the exact solutions.

1. Let us examine the problem of irrotational ideal fluid flow around a body apex or a wing angular point with half-angle θ at the apex. Let us introduce a Cartesian rectangular x, y, z coordinate system with x axis directed along the line of body (wing) symmetry, z axis in the plane of the wing (in the case of a cone, arbitrarily but perpendicular to the x axis), and y axis perpendicular to the x and z axes. The potential of the flow being investigated should satisfy the three-dimensional Laplace equation with boundary conditions of nonpenetration on the body (wing) surface. By virtue of the boundary conditions the problem is self-similar and, following [1-3], we seek its solution in the form

$$\Phi = Cx^n \varphi(y/x,\ z/x,\ \theta) \tag{1.1}$$

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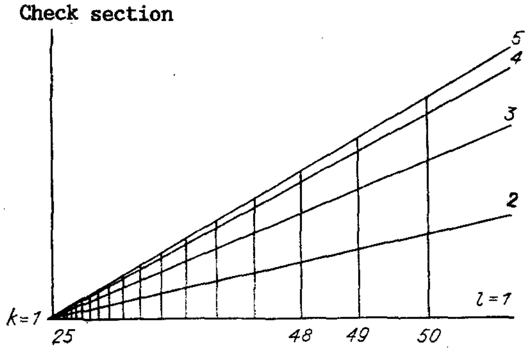

Fig. 1

[is a dimensionless function to be determined, C is a dimensional constant that is found only from the condition for merger with the solution of the external problem of the flow around a body (wing) of finite dimensions].

Since the dependence on the longitudinal coordinate is given by Eq. (1.1), it is sufficient to pose the nonpenetration condition only in one arbitrary body (wing) transverse section. The Neumann problem for the Laplace equation in the case of a wing angular point was solved in [1-3] by the method of separation of variables in a spherical coordinate system. As is shown in [4, 5], the problem under investigation reduces to the solution of a linear singular integral equation. In this case by virtue of the boundary conditions this equation is homogeneous, C in Eq. (1.1) drops out of the subsequent considerations, and nontrivial solutions exist only for certain n (the eigenvalue problem). Let us note that the desired self-similar solution is also an asymptotic of the solution of broader classes of nondegenerate nonself-similar problems about the flow around bodies of finite dimensions [6].

The methods of "vortex frames" (in the case of a wing angular point) or a panel method (in the case of a body apex) that result in a linear homogeneous system of equations were used for the numerical solution of the integral equation. The condition of degeneracy of this system determines the unknown quantity n in Eq. (1.1). This problem is a representative of the class of eigenvalue problems that was not solved earlier by the method of "discrete vortices," "vortex frames," or the panel method. For instance, the problem is not mentioned in [4, 7], where a detailed analysis is contained of the formulation of problems solved by the "discrete vortices" method.

The solution in the neighborhood of the apex in investigations of the flow around bodies and wings of finite dimensions is representable in the form of the sum of the eigenfunctions obtained multiplied by dimensional constants related to the characteristic dimensional quantities of the complete inhomogeneous problem.

2. Let us consider the problem of fluid flow in the neighborhood of a wing angular point. The flow potential can be represented in the form of a double-layer potential, where the condition of nonpenetration on the wing surface y = 0 (see [4, 5]) is

$$\frac{1}{4\pi}\int\limits_S \frac{\Gamma(\xi,\zeta)\,d\xi\,d\zeta}{[(\xi-x)^2+(\zeta-z)^2]^{3/2}}=0, \tag{2.1}$$

where $\Gamma(\xi, \zeta)$ is the intensity of the double-layer potential jump, and S is part of the plane $y = 0$; $x > 0$; $|z/x| < \lambda$; $\lambda = \tan\theta$. By virtue of assumption (1.1)

$$\Gamma(\xi,\zeta)=\xi^n\gamma(\zeta/\xi). \tag{2.2}$$

Taking Eq. (2.2) into account, Eq. (2.1) results in a linear homogeneous singular integral equation in γ. In order for this equation to have a nontrivial solution it is necessary to formulate the condition for its degeneracy, from which the unknown quantity n is indeed found.

Let us separate the whole domain S into a set of quadrangles $s_{k,\ell}$ by using a mesh formed by rays emerging from the origin and by the lines x = const (Fig. 1). If we assume $\Gamma(\xi, \zeta) = \Gamma_{k,\ell} = \text{const}$ for $\xi, \zeta \in s_{k,\ell}$, then following [4] the method of "vortex frames" can be utilized for the numerical solution of Eq. (2.1) where the circulation of the vortex

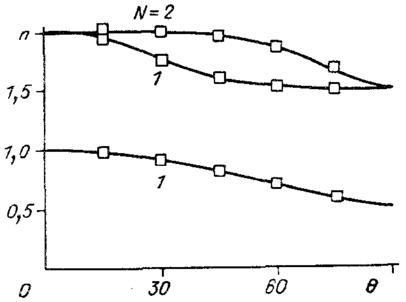

Fig. 2

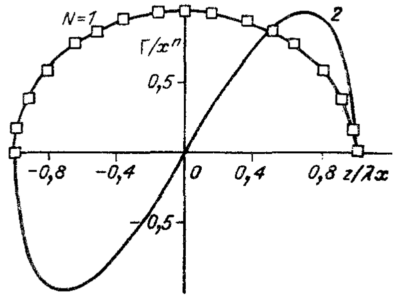

Fig. 3

frame located on the boundary of $s_{k,\ell}$ will be $\Gamma_{k,\ell}$. Since the transverse wing dimension increases linearly with the growth of x, a nonuniform partition was selected along the x axis. Along the z axis the mesh was constructed according to the cosine law. Therefore, the coordinates of the nodal points were selected as follows:

$$x_k = e^{-x_0+k\Delta x},\ z_{k,l} = -\lambda x_k \cos(\pi l/(L+1)),\ k = 1, \ldots, K,\ l = 1, \ldots, L \tag{2.3}$$

(x_0, Δx, K, L are given numbers). The check point coordinates are expressed analogously

$$x^* = e^{-x_0+(k_*+0,5)\Delta x},\ z_l^* = -\lambda x^* \cos(\pi(l-0.5)/L),\ l = 1, \ldots, L, \tag{2.4}$$

where k_* is a certain given number, $k_* < K$. Used in the computations were the parameters: K = 50, L = 10, $k_* = 25$, $\Delta x = \lambda/K$, $x_0 = -5$.

Thus, the panel $s_{k,\ell}$ is a quadrangle with coordinates (x_k, $z_{k,\ell+1}$), (x_{k+1}, $z_{k+1,\ell+1}$), (x_{k+1}, $z_{k+1,\ell}$), and (x_k, $z_{k,\ell}$), k = 1, ..., K, ℓ = 1, ..., L. The exception is the last panel on each ray $s_{K,\ell}$, for which $x_{K+\ell} = \infty$ (see Fig. 1). Then in conformity with Eq. (2.2)

$$\Gamma_{k,l} = G_l(x_k^*)^n,\ l = 1, \ldots, L,\ k = 1, \ldots, K-1, \tag{2.5}$$
$$x_k^* = e^{-x_0+(k+0,5)\Delta x}$$

along the ray. In the last panel s_K it is assumed

$$\Gamma_{K,l} = G_l x^n,\ x > x_K,\ l = 1, \ldots, L. \tag{2.6}$$

The condition of nonpenetration in the finite-difference approximation (2.3)-(2.6) takes the form

$$\sum_{l=1}^{L} A_{m,l}G_l = 0,\ m = 1, \ldots, L,\ A_{m,l} = \sum_{k=1}^{K} V_{m,k,l}(x_k^*)^{n-1} + V_{m,l}^{\mathrm{ac}}, \tag{2.7}$$

where $V_{m,k,\ell}$ is the vertical velocity component from a "vortex frame" of unit intensity (the panel $s_{k,\ell}$) to the check point x^*, z_m^*, and $V_{m,\ell}^{as}$ is the asymptotic estimate of the of the integral in Eq. (2.1) along the panel $s_{k,\ell}$, which is obtained by using a series expansion of the integrand in Eq. (2.1) in the small parameter x^*/ξ, $\xi \in s_{K,\ell}$ with Eq. (2.6). taken into account. Let the panel $s_{K,\ell}$ be formed by the rays $z = \lambda_\ell x$, $z = \lambda_{\ell+1}x$, $|\lambda_{\ell+1} - \lambda_\ell| \ll 1$ as well as the line $x = x_K$. Then we obtain the estimate

$$V_{m,l}^{\mathrm{as}} = \frac{(x_K)^{n-1}}{4\pi}\frac{\lambda_{l+1}-\lambda_l}{(1+\lambda_l^2)^{3/2}}\left[\frac{1}{n-1} - \frac{3B}{2(2-n)}\frac{x^*}{x_K} + \left(\frac{15}{8}B^2 - \frac{3}{2}\right)\frac{1}{3-n} \times \right. \tag{2.8}$$
$$\left. \times \left(\frac{x^*}{x_K}\right)^2 + \ldots\right],\ B = -\frac{2(1+\lambda_m^*\lambda_l)}{1+\lambda_l^2},\ \lambda_m^* = z_{K,m}^*/x_K,\ \lambda_l = z_{K,l}/x_K,$$

for the velocity $V_{m,\ell}^{as}$ at the check point with number m. Thus, we arrive at a system of linear equations (2.7) with zero right side. Analysis of the degeneracy of this system, performed numerically, permits determination of the unknown eigenvalue n of the problem as well as finding the eigenfunction $\Gamma_{k,\ell}$ from Eq. (2.5).

It should be noted that insertion of $V_{m,\ell}^{as}$ in Eq. (2.7) is essential since very large values of K unacceptable for an electronic computer (for a fixed number of the check section k_*) must be taken if it is not present. Three terms in $V_{m,\ell}^{as}$ were retained in the computations. As n increases the quantity of terms in the expansion (2.8) should also grow.

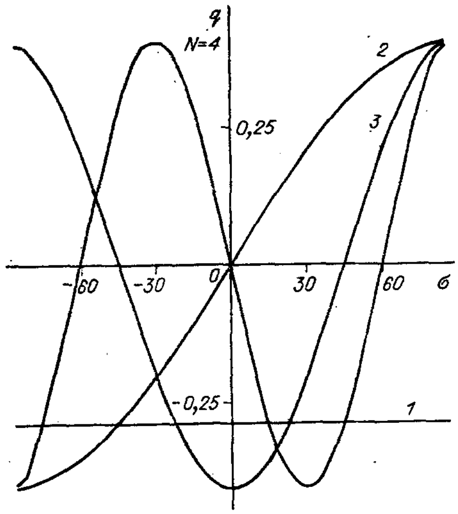

Fig. 4

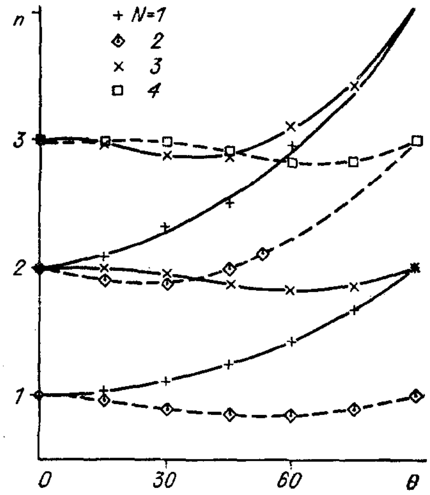

Fig. 5

Results of computing the eigenvalues and eigenfunctions in the range $0 < n < 2$ are displayed in Figs. 2 and 3 in comparison with the results of [2, 3] (the solid lines are from this paper, the squares are the numerical method in [3], and N is the number of the eigenfunction). The agreement can be acknowledged good throughout.

3. Let us investigate the problem of ideal fluid flow in the neighborhood of the apex of a circular cone. For its numerical solution we partition the cone surface for $x \leq x_K$ into quadrangular panels by the planes $x = \text{const}$ and $\sigma = \text{const}$ ($\tan\sigma = y/z$). We select the partition along the angle σ to be uniform and the distance between the sections along x such that $R_{k+1} - R_k = [(\pi R_{k+1})/2L] \sin\theta$, $R_K = 10$, $R_{k+1} - R_k \leq R_k$ (R is the distance from the cone apex, L is the quantity of partitions along the angle in a quarter cone, and K along the x axis, while $k = 1, \ldots, K$). For $R > R_K$ we have 4L semi-infinite panels. Let us introduce a source distribution on the panel surfaces. Then the nonpenetration condition can be written in the form

$$(\mathbf{V}\cdot\mathbf{n}) = 0, \tag{3.1}$$

where

$$\mathbf{V} = \frac{1}{4\pi}\int\int\frac{Q\,[(x-\xi)\,\mathbf{i} + (y-\eta)\,\mathbf{j} + (z-\zeta)\,\mathbf{k}]\,ds}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}};$$

$\mathbf{n} = \sin\theta\mathbf{i} - \cos\theta\sin\sigma\mathbf{j} - \cos\theta\cos\sigma\mathbf{k}$ is the inner normal, $Q(x, \sigma) = R^{n-1}q(\sigma)$ is the source distribution density. We arrange the check points in the sections $k = k_*$ at the points $R^* = (R_k + R_{k+1})/2$, $k = k_*$, $y_\ell^* = R^* \sin\theta \sin\sigma_\ell^*$, $z_\ell^* = R^* \sin\theta \cos\sigma_\ell^*$, $\sigma_\ell^* = [\pi(2\ell - 1)/4L]$, $\ell = 1, \ldots, L$. The source distribution density is constant along the panel in the computations

$$Q(x, \sigma) = (R_k^*)^{n-1} q_l \text{ for } x \leqslant x_K,$$
$$R_k^* = (R_k + R_{k+1})/2,\ k = 1, \ldots, K. \tag{3.2}$$

Substituting Eq. (3.2) into Eq. (3.1), we obtain a system of equations of the type (2.7):

$$\sum_{l=1}^{L} A_{m,l} q_l = 0,\ m = 1, \ldots, L,\ A_{m,l} = \sum_{k=1}^{K} V_{m,k,l}(R_k^*)^{n-1} + V_{m,l}^{\mathbf{as}}.$$

Here $V_{m,k,\ell}$ is the velocity component normal to the cone surface at the point $x^* = R^* \cos\theta$, y_m^*, z_m^*, induced by the quadrangular panel with subscripts k, ℓ and source density one; $V_{m,\ell}^{as}$ is the asymptotic estimate for small R^*/R of the normal velocity component induced by a narrow semi-infinite panel of sources ($R > R_K$) whose density distribution changes as R^n. If the contributions to the normal velocity component from four such panels, located at $\arctan(y/z) = \sigma, -\sigma, \pi - \sigma, \pi + \sigma$, are summed then we have for three terms of the $V_{m,\ell}^{as}$ expansion

$$V_{m,l}^{\mathbf{as}} = \frac{q_l}{2L}\sin\sigma_l^*\sin\sigma_m^*\sin^2\theta\cos\theta\, R_K^{n-1}\Big\{-\frac{1}{n-1} - \frac{3\cos 2\theta}{n-2}\frac{R^*}{R_K} +$$

$$+\left[\frac{3}{2}-\frac{15}{2}\cos^4\theta+\frac{15}{2}\sin^2 2\theta-15\sin^4\theta\cos^2\sigma_l^*\cos^2\sigma_m^*-\right.$$
$$\left.-\frac{15}{2}\sin^2\theta\left(\cos^2\sigma_l^*\cos^2\sigma_m^*+\sin^2\sigma_l^*\sin^2\sigma_m^*\right)\right]\frac{1}{n-3}\left(\frac{R^*}{R_K}\right)^2+\ldots\Bigg\}$$

for $q(\sigma) = -q(-\sigma)$ or

$$V_{m,l}^{\mathbf{as}}=\frac{q_l}{2L}\sin^2\theta\cos\theta R_K^{n-1}\left\{\frac{1}{n-1}+\frac{3}{n-2}\left[\cos^2\theta-\sin^2\theta\times\right.\right.$$
$$\left.\times\left(\cos^2\sigma_l^*\cos^2\sigma_m^*+\sin^2\sigma_l^*\sin^2\sigma_m^*\right)\right]\frac{R^*}{R_K}-\frac{3}{2(n-3)}\left(\frac{R^*}{R_K}\right)^2\times$$
$$\left.\times\left[1-5\cos^4\theta-5\sin^2\theta(1-3\cos^2\theta)(\cos^2\sigma_l^*\cos^2\sigma_m^*+\sin^2\sigma_l^*\sin^2\sigma_m^*)\right]+\ldots\right\}$$

for $q(\sigma) = q(-\sigma)$.

The computed eigenfunctions are shown in Fig. 4 for different n (N is the number of the eigenfunction).

The problem about the bow around the apex of a circular cone can be reduced to the solution of an ordinary differential equation. Let us write the Laplace equation in spherical coordinates R, β, σ ($x = R\cos\beta$, $y = R\sin\beta\sin\sigma$, $z = R\sin\beta\cos\sigma$)

$$\frac{\partial^2\Phi}{\partial R^2}+\frac{2}{R}\frac{\partial\Phi}{\partial R}+\frac{1}{R^2}\frac{\partial^2\Phi}{\partial\beta^2}+\frac{\operatorname{ctg}\beta}{R^2}\frac{\partial\Phi}{\partial\beta}+\frac{1}{R^2\sin^2\beta}\frac{\partial^2\Phi}{\partial\sigma^2}=0.$$

Substituting a potential in the form $\Phi = CR^n h(\sigma)f(\beta)$ into this equation, we obtain

$$-\frac{\sin^2\beta}{f}\left[n(n+1)f+\operatorname{ctg}\beta f'+f''\right]=\frac{h''}{h}=p^2$$

(p = const). By virtue of the periodicity of the potential in σ, we have $h(\sigma) = \sin(p\sigma + \Delta\sigma)$, $p = 0, 1, \ldots$, where $\Delta\sigma = 0$ for $\Phi(\sigma) = -\Phi(-\sigma)$, $\Delta\sigma = \pi/2$ for $\Phi(\sigma) = \Phi(-\sigma)$. We write the following equation for the function f

$$f''+\operatorname{ctg}\beta f'+[n(n+1)-p^2/\sin^2\beta]f=0. \tag{3.3}$$

The boundary conditions for Eq. (3.3) are the nonpenetration conditions on the cone surface $f'(\theta) = 0$ and boundedness of the potential on the x axis for $x < 0$ [$|f(\pi)| < \infty$]. In general, only the function $f(\beta) = 0$ satisfies Eq. (3.3) with the boundary conditions mentioned and only for certain n are there nontrivial solutions (the eigenvalue problem). The solution of this problem can be obtained numerically with high accuracy.

A comparison between the eigennumbers computed by the panel method (markers) and the dependences $n(\theta)$ obtained from the solution of Eq. (3.3) is presented in Fig. 5. Good agreement between the results is observed in the range of numbers n investigated.

4. An important aspect of the application of the proposed method is the construction of substantially three-dimensional test-problems to estimate the accuracy of numerical methods (the "vortex frame" method, the panel method, etc.) since there are exact solutions of the appropriate problems (see [3] and Sec. 3 of this paper) in two particular cases (the apex of a delta wind and a circular cone).

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INTERACTION OF PLANE NON-PARALLEL JETS

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UDC 532.526

The collision of near-wall jets on a smooth surface was examined by many authors [1, 2]. Within the framework of the theory of potential jet flows, the position of the interaction domain remains undetermined in this problem while the direction of the resultant jet is determined uniquely. Taking account of viscosity permits finding the position of the interaction domain [3]. It is important to note that integral conservation laws and the assumption that viscosity is not essential in the interaction domain are sufficient to establish the direction of the resultant jet. It is not necessary to know the pressure distribution on the surface here.

A feature of the problem of jet collision in the neighborhood of a corner is the fact that the integral conservation laws do not permit establishment of the direction of the resultant jet if the pressure distribution on the surface is not known in the interaction domain. The solution of this problem within the framework of the theory of potential jet flows is also not unique and does not permit the unique determination of the resultant jet direction.

A simple approximation solution of the problem of the collision of plane submerged incompressible near-wall jets in the neighborhood of a corner is represented in this paper within the framework of the infinitely thin jet approximation. Laminar and turbulent flow are examined in a quasilaminar approximation. It is noted that small changes in the interacting jet parameters can radically alter the resultant jet direction. The expediency is also shown of utilizing the infinitely thin jet approximation in other jet flow problems. A qualitative examination is performed of problems about jet impact in a corner and on the collision of several jets in space.

1. Let two plane near-wall submerged jets directed toward the line of plane intersection be propagated along two intersecting planes Ω_1 and Ω_2 (Fig. 1). The angle γ between the planes and the coordinates x_1 and x_2 are taken along the surfaces Ω_1 and Ω_2, respectively, along the normals to the line of intersection on which $x_1 = 0$, $x_2 = 0$. Let us assume that the domain of jet interaction lies near a point with coordinates $x_1 = 0$, $x_2 = 0$ and that the jet parameters at a certain distance from the corner are independent of the flow in the interaction domain. It is assumed that the jet sources are sufficiently remote from the interaction domain. We assume that the flow therein is stationary and has the following configuration: near the corner each of the jets is separated from the surface ($x_1{}^0$ and $x_2{}^0$ are the coordinates of the point of jet separation on the surfaces Ω_1 and Ω_2), the flow in front of the separation point is unperturbed, behind the separation point a domain is shaped with small changes in the pressure and low velocities which is considered stagnant, and one resultant jet is formed because of the collision. Assuming the jet parameters outside the interaction domain known for $x_1 > x_1{}^0$ and $x_2 > x_2{}^0$, we determine the resultant jet direction, the pressure in the stagnant zone, and its characteristic dimensions.

The question of the motion of the jet being separated must be examined to solve the formulated problem. Later the jet motion after separation is investigated in the infinitely thin jet (ITJ) approximation.

2. Let ξ be a vector line of the momentum field and outside this line the momentum equals zero. Let us write the momentum conservation law in the ξ, τ coordinate system

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